



## Tying Knots in Light Fields

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We construct analytically, a new family of null solutions to Maxwell's equations in free space whose field lines encode all torus knots and links. The evolution of these null fields, analogous to a compressible flow along the Poynting vector that is shear free, preserves the topology of the knots and links. Our approach combines the construction of null fields with complex polynomials on  $S^3$ . We examine and illustrate the geometry and evolution of the solutions, making manifest the structure of nested knotted tori filled by the field lines.

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Knots and the application of mathematical knot theory to space-filling fields are enriching our understanding of a variety of physical phenomena with examples in fluid dynamics [1–3], statistical mechanics [4], and quantum field theory [5], to cite a few. Knotted structures embedded in physical fields, previously only imagined in theoretical proposals such as Lord Kelvin's vortex atom hypothesis [6], have in recent years become experimentally accessible in a variety of physical systems, for example, in the vortex lines of a fluid [7–9], the topological defect lines in liquid crystals [10,11], singular lines of optical fields [12], magnetic field lines in electromagnetic fields [13–15], and in spinor Bose-Einstein condensates [16]. Furthermore, numerical simulations have shown that stable knotlike structures arise in the Skyrme-Faddeev model [17,18], and consequently in triplet superconductors [19,20] and charged Bose condensates [21]. Analytical solutions for such excitations, however, are difficult to construct owing to the inherent nonlinearity in most dynamical fields and have therefore remained elusive.

An exception is a particularly elegant solution to Maxwell's equations in free space (see Fig. 1), brought to light by Rañada [22], which provides an encouraging manifestation of a persistent nontrivial topological structure in a linear field theory. This solution, referred to as the Hopfion solution for the rest of the Letter, can furthermore be experimentally realized using tightly focused Laguerre-Gaussian beams [14].

In this Letter, we present the first example of a family of exact knotted solutions to Maxwell's equations in free space, with the electric and magnetic field lines encoding all torus knots and links, which persist for all time. The unique combination of experimental potential and opportunity for analytical study makes light an ideal candidate for studying knotted field configurations and furthermore, a means of potentially transferring knottedness to matter.

In the case of the Hopfion solution illustrated in Fig. 1, the electric, magnetic, and Poynting field lines exhibit a

remarkable structure known as a Hopf fibration, with each field line forming a closed loop such that any two loops are linked. At time  $t = 0$ , each of the electric, magnetic, and Poynting field lines have identical structure (that of a Hopf fibration), oriented in space so that they are mutually orthogonal to each other. The topology of these structures is preserved with time, as the electric and magnetic field lines evolve like unbreakable filaments embedded in a fluid flow, stretching and deforming while retaining their identity [15,23]. The Poynting field lines evolve instead via a rigid translation along the  $z$  axis. The Hopfion solution has been rediscovered and studied in several contexts [14,22,24–27] and can be constructed in many ways using complex scalar maps, spinors, twistors.

Despite numerous attempts at generalizing the Hopfion solution to light fields encoding more complex knots, the problem of constructing light fields encoding knots that are preserved in time has remained open until now. Attempts at generalizing Hopfions to torus knots [14,15,28] succeeded at constructing such solutions at an instant in time, but their structure was not preserved [15], and unraveled with time. Beyond Maxwell's equations, the more general problem of finding explicit solutions to dynamical flows which embody persistent knots has also remained open.

The fluidlike topology-preserving evolution of the Hopfion solution is closely tied to the property that the electric and magnetic fields are everywhere perpendicular and of equal magnitude [a constraint known as nullness, cf. Eq. (3)]. Nullness introduces an effective nonlinearity in the problem and imposes a dynamical geometric constraint on Maxwell fields, restricting the space of possible topological configurations of field lines.

We construct knotted solutions within the space of null field configurations by making use of formalisms developed for the construction of null Maxwell fields, such as Bateman's method [29] or equivalently a spinor formalism (see Supplemental Material [30]). The combination of a null electromagnetic field formalism with a topological

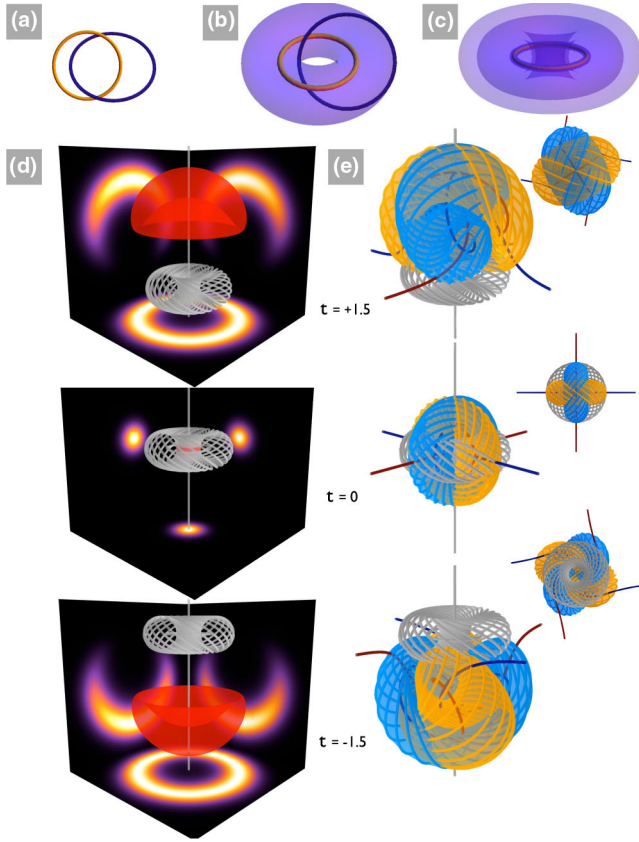


FIG. 1 (color online). Hopfion solution: field line structure (a)–(c) and time evolution (d)–(e). Field lines fill nested tori, forming closed loops linked with every other loop. (a) Hopf link formed by the circle at the core (orange) of the nested tori, and one of the field lines (blue). (b) The torus (purple) that the field line forming the Hopf link is tangent to. (c) Nested tori (purple) enclosing the core, on which the field lines lie. (d) Time evolution of the Poynting field lines (gray), an energy isosurface (red), and the energy density (shown via projections). (e) Time evolution of the electric (yellow), magnetic (blue), and Poynting field lines (gray), with the top view shown in the inset.

construction, leading to a family of knotted null solutions is the central result of this Letter. We now briefly review the key features of the evolution of null electromagnetic fields.

*Null electromagnetic fields.*—Null electromagnetic fields have a rich history, from the early construction by Bateman [29] to Robinson’s theorem [31] and Penrose’s twistor theory [32]. For a null electromagnetic field, the Poynting field not only guides the flow of energy, but also governs the evolution of the electric and magnetic field lines. These field lines evolve as though embedded in a fluid, flowing at the speed of light, in the direction of the Poynting field [15,23]. The persistence of the null conditions guarantees the continued fluidlike evolution of the electric and magnetic field lines, giving them the appearance of unbreakable elastic filaments.

The preservation of the null conditions requires the continued perpendicularity of the electric and magnetic

fields as they evolve, thus requiring the flow transporting the field lines to be free of shear. Robinson’s theorem [31] guarantees the existence of a shear-free family of light rays associated with every null electromagnetic field. In flat space-time, this shear-free family of light rays is given by the normalized Poynting field: the velocity field of the flow transporting the field lines.

The Hopfion solution illustrated in Fig. 1 beautifully demonstrates the features of a null electromagnetic field: the electric and magnetic field lines evolve smoothly, preserving the topology of the field line structure (a Hopf fibration in both cases). The shear-free family of light rays associated with the Hopfion solution, remarkably, also has the structure of a Hopf fibration, which remains unchanged as it evolves with time. This family of light rays is well known in the literature as the Robinson congruence [32].

Since the null condition makes the design of a knotted magnetic or electric field, a problem of engineering a triplet of mutually orthogonal fields that remains orthogonal under time evolution, we start with the formalisms developed for the construction of null fields and seek to construct knotted structures within them. We now briefly summarize Bateman’s method for constructing null electromagnetic fields.

*Bateman’s construction.*—Bateman [29] constructs all null electromagnetic fields associated with the same underlying normalized Poynting field, using two complex scalar functions of space-time. Hogan [33], has shown that all null electromagnetic fields can be constructed using Bateman’s method.

According to Bateman’s construction, given a pair of complex scalar functions of space-time ( $\alpha, \beta$ ) which satisfy

$$\nabla \alpha \times \nabla \beta = i(\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha), \quad (1)$$

there is a corresponding electromagnetic field

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \nabla \alpha \times \nabla \beta, \quad (2)$$

where  $\mathbf{F}$  is known as the Riemann-Silberstein vector [34]. This field is null (both invariants vanish),

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad \mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B} = 0, \quad (3)$$

since the scalar product  $\mathbf{F} \cdot \mathbf{F}$  is zero, as can be seen by taking the dot product of the left-hand side of Eq. (1) with its right-hand side. For the null solutions generated by Eq. (2) to be nontrivial, the following conditions must be satisfied:  $\partial_t \alpha ((\partial_t \alpha)^2 - (\nabla \alpha)^2) = \partial_t \beta ((\partial_t \beta)^2 - (\nabla \beta)^2) = 0$

Each pair ( $\alpha, \beta$ ) satisfying Eq. (1) generates a whole family of fields because any vector field of the form

$$\mathbf{F} = h(\alpha, \beta) \nabla \alpha \times \nabla \beta = \nabla f(\alpha, \beta) \times \nabla g(\alpha, \beta), \quad (4)$$

where  $h := \partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g$  and  $f, g$  are arbitrary holomorphic functions of  $(\alpha, \beta)$ , is a null electromagnetic field. Note that all fields constructed in this way have, by construction, the same normalized Poynting field:  $\mathbf{E} \times \mathbf{B} / |\mathbf{E} \times \mathbf{B}| = i(\mathbf{F} \times \mathbf{F}^* / \mathbf{F} \cdot \mathbf{F}^*)$ , where  $\mathbf{F}^*$  is the

complex conjugate of  $\mathbf{F}$ . This is made manifest, when these null fields are expressed in the equivalent language of spinors (see Supplemental Material [30]).

We list here two simple examples of this construction [27]: a circularly polarized plane wave traveling in the  $+z$  direction and the Hopfion solution. They arise from the following choices of  $\alpha$  and  $\beta$ . For the plane wave:  $\alpha = z - t$ ,  $\beta = x + iy$ ,  $f = e^{i\alpha}$ ,  $g = \beta$ , giving  $\mathbf{F}^{\text{pw}} = (\hat{x} + i\hat{y})e^{i(z-t)}$ . For the Hopfion we have instead  $\alpha = -d/b$ ,  $\beta = -ia/(2b)$ ,  $f = 1/\alpha^2$ ,  $g = \beta$  giving  $\mathbf{F}^{\text{hp}} = d^{-3}(b^2 - a^2, -i(a^2 + b^2), 2ab)$ , where  $a = x - iy$ ,  $b = t - i - z$ ,  $d = r^2 - (t - i)^2$ .

We now present a family of light-beam-like propagating solutions to Maxwell's equations in free space, in which the electric and magnetic fields encode torus knots and links that are preserved in time. We construct these solutions using complex scalar maps in the context of Bateman's framework. We then describe the knotted structure of the field lines, and compute the entire set of conserved currents, the helicity and charges for electromagnetism in free space, for this family of solutions.

*Constructing knotted null electromagnetic fields.*— There is a natural connection between knots and singular points of complex maps from  $\mathbb{S}^3$  to  $\mathbb{C}$ . This was used, for example, in recent work by Dennis *et al.* [12] to construct knotted optical vortices in light beams. In particular, given a pair of complex numbers  $(u, v)$  such that  $|u|^2 + |v|^2 = 1$  (and hence they define coordinates on  $\mathbb{S}^3$ ), it has been shown [35,36] that  $u^p \pm v^q = 0$  is the equation of a  $(p, q)$  torus knot, when  $(p, q)$  are coprime integers.

We note that the following choice of  $(\alpha, \beta)$  in Bateman's construction:

$$\alpha = \frac{r^2 - t^2 - 1 + 2iz}{r^2 - (t - i)^2}, \quad \beta = \frac{2(x - iy)}{r^2 - (t - i)^2}, \quad (5)$$

which satisfies Eq. (1), admits a natural interpretation as coordinates on  $\mathbb{S}^3$  since  $|\alpha|^2 + |\beta|^2 = 1$  for any  $t$ . At  $t = 0$ ,  $(\alpha, \beta) = (u, v)$ , the standard stereographic coordinates on  $\mathbb{S}^3$ .

Hence by [35,36],  $\alpha^p \pm \beta^q = 0$  encodes a singular line tied into a  $(p, q)$  torus knot when  $p, q$  are coprime integers. Guided by this result, we make the intuitive choice of  $f(\alpha, \beta) = \alpha^p$  and  $g(\alpha, \beta) = \beta^q$  in (4), to obtain the following family of knotted null solutions:

$$\mathbf{F} = \nabla\alpha^p \times \nabla\beta^q, \quad (6)$$

which can equivalently be expressed in terms of spinors (see Supplemental Material [30]). On inspection, we find that the electric and magnetic field lines (shown in Fig. 2) are grouped into knotted and linked tori, nested one inside the other, with  $(p, q)$ -torus knots at the core of the foliation. Being smooth, finite energy solutions to Maxwell's equations, these knotted electromagnetic fields are physically feasible, and nullness guarantees that the topology of these

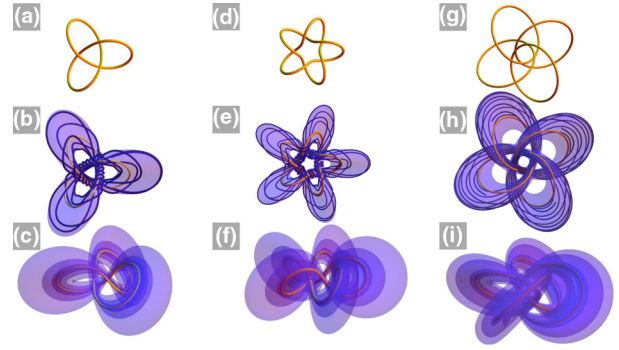


FIG. 2 (color online). Structure of magnetic field lines, (a)–(c) Trefoil knots ( $p = 2$ ,  $q = 3$ ), (d)–(f) Cinquefoil knots ( $p = 2$ ,  $q = 5$ ), (g)–(i) 4 linked rings ( $p = 2$ ,  $q = 2$ ). (a),(d),(g) Core (orange) field line(s) forming (a) a trefoil knot (d) a cinquefoil knot (g) 4 linked rings. (b),(e),(h) Field line(s) (blue) wrapping around the core (orange) confined to a knotted torus (purple) enclosing the core. (c),(f),(i) Knotted nested tori (purple) enclosing the core, on which the field lines lie.

knotted structures is preserved in time (as shown in Fig. 3). The shear-free family of light rays associated with this family of solutions is the Robinson congruence.

As illustrated in Fig. 2, the magnetic field lines organize around a set of core magnetic field lines, which form  $(p, q)$ -torus knots, and stay confined on the surfaces of nested tori, which are isosurfaces of  $\Psi_B = \text{Re}\{\alpha^p \beta^q\}$ . The innermost core of these nested tori has zero thickness, and corresponds to the knotted core magnetic field lines. Starting from the core, the tori successively increase in thickness (as shown in Fig. 2), until they collide (when  $\Psi_B = 0$ ) and extend to infinity. Since the magnetic field is divergence free and does not vanish on any isosurface  $\Psi_B \neq 0$ , it follows [37] that all magnetic lines are either periodic or quasiperiodic on each toroidal surface.

As the field evolves, the nested tori along with the knotted core deform smoothly, rotating and stretching, as illustrated in Fig. 3, and the supplementary videos [30].

The electric field lines are also confined on the surface of nested tori (isosurfaces of  $\Psi_E = \text{Im}\{\alpha^p \beta^q\}$ ), organizing around a set of knotted core electric field lines, and have exactly the same structure as the magnetic field lines, rotated in space about the  $z$  axis by  $\pi/(2q)$ . For more detailed descriptions and explicit equations describing the core field lines, see Supplemental Material [30].

To further characterize the physical properties of this family of knotted null fields, we compute the helicity and the full set of conserved quantities [14] corresponding to the known (conformal) symmetries of electromagnetism in free space. The nonvanishing currents and charges normalized by the energy are the following: magnetic helicity  $\mathcal{H}_m =$  electric helicity  $\mathcal{H}_e = 1/(p + q)$ , momentum  $\mathbf{P} = \mathbf{J}_{\text{SCT}} = (0, 0, (-p)/(p + q))$ , angular momentum  $\mathbf{L} = (0, 0, q/(p + q))$ , where  $\mathbf{J}_{\text{SCT}}$  is the current associated with special conformal transformations; see Supplemental Material [30] for explicit expressions.

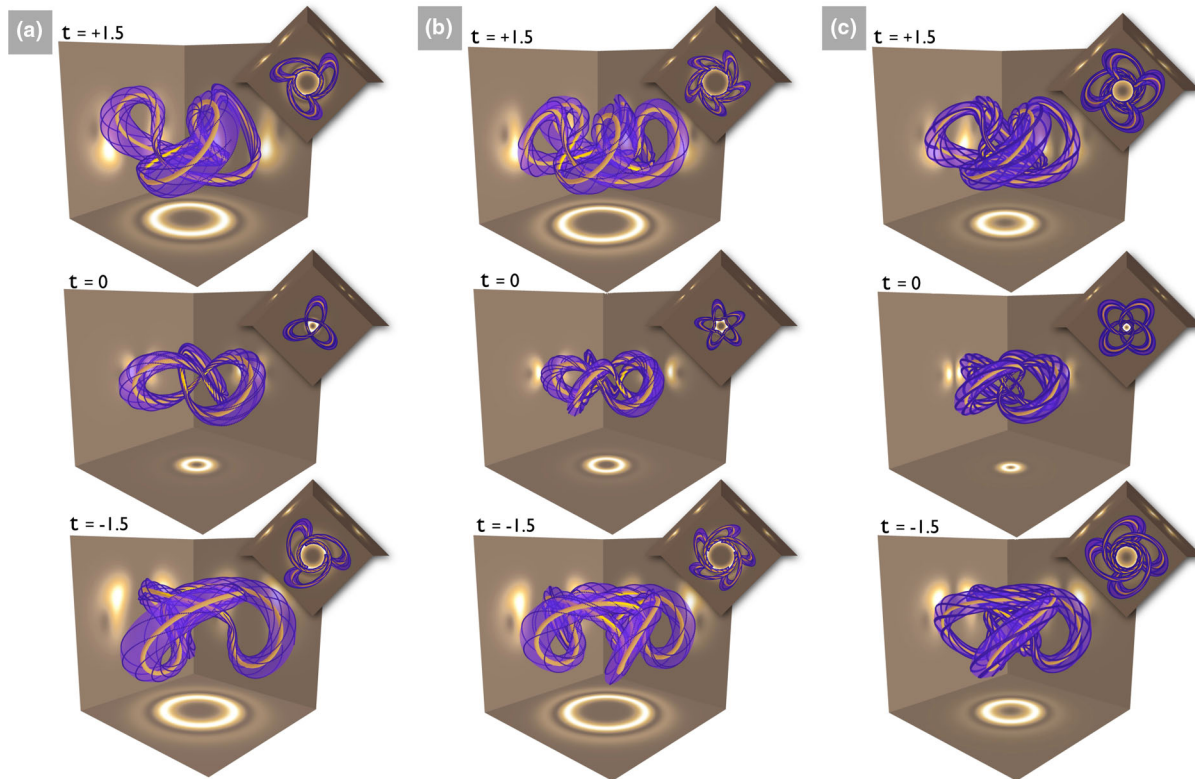


FIG. 3 (color online). Time evolution of magnetic field lines and energy density for (a) the trefoil knot ( $p = 2$ ,  $q = 3$ ), (b) the cinquefoil knot ( $p = 2$ ,  $q = 5$ ), and (c) the 4 Hopf-linked rings ( $p = 2$ ;  $q = 2$ ). Shown is the topology preserving, fluidlike evolution of the core field line(s) (orange), and field line(s) (blue) lying on tori (purple) enclosing the core.

The special choice of  $(p, q) = (1, 1)$  yields the Hopfion solution described earlier, in which not only do the core electric and magnetic field lines form Hopf links, but all the other field lines are also closed loops, linked with every other field line.

*Summary.*—The solutions presented here extend the space of exact, physically feasible, knotted Maxwell fields beyond the Hopfion, by encoding an entire family of both knots and links that are preserved under time evolution. Many open questions remain on the space of knotted states, such as whether solutions with each and every field line knotted and preserved by time evolution exist with topology different from the Hopf fibration (e.g., a Seifert foliation of  $\mathbb{S}^3$ ). Beyond electromagnetism, it remains an open question whether similar explicit solutions can be found for nonlinear evolutions such as the Euler flow of ideal fluids. From a dynamical systems perspective, it may be interesting to explore the role of the invariant tori in the solutions we present and the conditions for which Bateman’s construction give rise to electric and magnetic fields with a first integral. Finally, if realized in experiment, can these structures be imprinted on matter such as plasmas or quantum fluids?

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